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Statistical mechanics of fluid turbulence  
taking into account fluid velocity

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### I. Introduction

The research on fluid turbulence has its long history. J. Boussinesq first considered the transport coefficient of fluid turbulence as

$$-\rho \overline{u_i u_j} = \rho K \frac{\partial u_i}{\partial x_j}, \quad (1)$$

where  $x_j$  is  $j$  component of space and  $u_i$  is  $i$  component of fluid velocity.  $\bar{a}$  is the average of quantity  $a$  with respect to space and is described as

$$\bar{a}(\underline{x}) = \int d\underline{x}_1 G(\underline{x}, \underline{x}_1) a(\underline{x}_1). \quad (2)$$

$G(\underline{x}, \underline{x}_1)$  is the kernel of integration and spatial average in this proceeding.  $K$  is the eddy viscosity coefficient and was referred in [1, 2].

$u_i$  is decomposed as

$$u_i = \overline{u_i} + u'_i, \quad (3)$$

where  $u_i$  is  $i$  component of fluid velocity at space  $\mathcal{V}$  and time  $t$ .

## II. Large-eddy simulation and its expansion

The transport equation of fluid turbulence is described as

$$\frac{\partial}{\partial t} U + \sum_{i=1}^3 \frac{\partial}{\partial x_i} F_i = 0, \quad (4)$$

where

$$U = \begin{pmatrix} \rho \\ \rho u_1 \\ \rho u_2 \\ \rho u_3 \\ \rho e \end{pmatrix} \quad (5)$$

and

$$F_i = \begin{pmatrix} \rho u_i \\ \rho u_i u_1 + p \delta_{i1} - 2\mu A_{i1} \\ \rho u_i u_2 + p \delta_{i2} - 2\mu A_{i2} \\ \rho u_i u_3 + p \delta_{i3} - 2\mu A_{i3} \\ (\rho e + p) u_i - 2\mu \sum_{j=1}^3 A_{ij} u_j - \lambda \frac{\partial T}{\partial x_i} \end{pmatrix} \quad (b)$$

$\rho$ ,  $u_i$ ,  $p$ ,  $\mu$ ,  $A_{ij}$ ,  $\lambda$ , and  $T$  are density,  $i$  component of velocity, pressure, molecular viscosity coefficient,  $ij$  element of stress tensor, thermal diffusion coefficient, and temperature of fluid at space  $\vec{x}$ , respectively.

$A_{ij}$  is described as

$$A_{ij} = \frac{1}{2} \left[ \frac{\partial u_j}{\partial x_i} + \frac{\partial u_i}{\partial x_j} - \frac{2}{3} (\vec{\nabla} \cdot \vec{u}) \delta_{ij} \right]. \quad (c)$$

Temperature  $T$  is defined in the relation as

$$\frac{3}{2} k_B T = \frac{1}{2} \sum_{j=1}^N \sum_{i=1}^3 m(j) (u(j)_i - \bar{u}_i)^2, \quad (d)$$

where

$$\bar{u}_i = \frac{\sum_{j=1}^N m(j) u(j)_i}{\sum_{j=1}^N m(j)} \quad (e)$$

and  $k_B$  is Boltzmann constant.

$T$  is defined at space  $\mathbf{x}$  over Kolmogorov length  $l_K$ .  $m(j)$  is mass of  $j$ th molecule and  $u(j)_i$  is  $i$  component of  $j$ th molecule's velocity.

Spatial average of eq. (4) is described as

$$\frac{\partial}{\partial t} \bar{U} + \sum_{i=1}^3 \frac{\partial}{\partial x_i} \bar{F}_i = 0, \quad (10)$$

where

$$\bar{U} = \begin{pmatrix} \bar{\rho} \\ \bar{\rho} \tilde{u}_1 \\ \bar{\rho} \tilde{u}_2 \\ \bar{\rho} \tilde{u}_3 \\ \bar{\rho} \tilde{e} \end{pmatrix} \quad (11)$$

and

$$\bar{F}_i = \begin{pmatrix} \bar{\rho} \tilde{u}_i \\ \bar{\rho} \tilde{u}_i \tilde{u}_1 + \bar{\omega} \delta_{i1} - 2\mu_t \bar{A}_{i1} \\ \bar{\rho} \tilde{u}_i \tilde{u}_2 + \bar{\omega} \delta_{i2} - 2\mu_t \bar{A}_{i2} \\ \bar{\rho} \tilde{u}_i \tilde{u}_3 + \bar{\omega} \delta_{i3} - 2\mu_t \bar{A}_{i3} \\ (\bar{\rho} \tilde{e} + \bar{\omega}) \tilde{u}_i - 2\mu \sum_{j=1}^3 A_{ij} u_j - \lambda_t \frac{\partial \rho}{\partial x_i} \end{pmatrix}. \quad (12)$$

Spatial average  $\bar{a}$  of quantity  $a$  at space  $\mathcal{E}$  is described as

$$\bar{a}(\underline{x}) = \int d\underline{x}_1 G(\underline{x}, \underline{x}_1) a(\underline{x}_1), \quad (13)$$

where  $G(\underline{x}, \underline{x}_1)$  is the kernel of integration.

$G(\underline{x}, \underline{x}_1)$  is the weight of spatial average and the weight is constant in this proceeding.

Eq. (12) is the spatial average of quantity  $a$  over space  $\mathcal{E}_1$ .  $\hat{b}$  is defined as

$$\hat{b} \equiv \frac{\overline{\rho b}}{\bar{\rho}}. \quad (14)$$

$\tilde{\omega}$  is defined as

$$\tilde{\omega} \equiv \bar{p} - \frac{1}{3} \sum_{\ell=1}^3 \Gamma_{\ell\ell}, \quad (15)$$

and  $\Gamma_{\ell m}$  is

$$\Gamma_{\ell m} = - \overline{\rho u_\ell u_m} + \bar{\rho} \tilde{u}_\ell \tilde{u}_m. \quad (16)$$

$\mathcal{D}$  is defined as

$$\mathcal{D} \equiv \hat{T} - \frac{1}{2\bar{\rho} c_v} \sum_{\ell=1}^3 \Gamma_{\ell\ell}. \quad (17)$$

$\mu_+$  and  $\lambda_+$  are eddy viscosity coefficient and thermal diffusion coefficient, respectively.

It is possible to expand eq. (10) as

$$\frac{\partial^2}{\partial t^2} \bar{\Psi} + \varepsilon A_{\bar{\Psi}} \frac{\partial}{\partial t} \bar{\Psi} = W_{\bar{\Psi}} \frac{\partial^2}{\partial x_1^2} \bar{\Psi}. \quad (18)$$

Spatial dimension is one for simplicity.

$A_{\bar{\Psi}}$  and  $W_{\bar{\Psi}}$  are matrices as

$$A_{\bar{\Psi}} = \begin{pmatrix} a_{\bar{\rho}} & 0 & 0 & 0 & 0 \\ 0 & a_{\bar{\rho}u_1} & 0 & 0 & 0 \\ 0 & 0 & a_{\bar{\rho}u_2} & 0 & 0 \\ 0 & 0 & 0 & a_{\bar{\rho}u_3} & 0 \\ 0 & 0 & 0 & 0 & a_{\bar{\rho}e} \end{pmatrix} \quad (19)$$

and

$$W_{\bar{\Psi}} = \begin{pmatrix} w_{\bar{\rho}}^2 & 0 & 0 & 0 & 0 \\ 0 & w_{\bar{\rho}u_1}^2 & 0 & 0 & 0 \\ 0 & 0 & w_{\bar{\rho}u_2}^2 & 0 & 0 \\ 0 & 0 & 0 & w_{\bar{\rho}u_3}^2 & 0 \\ 0 & 0 & 0 & 0 & w_{\bar{\rho}e}^2 \end{pmatrix}, \quad (20)$$

respectively.  $w_{\bar{\rho}}$ ,  $w_{\bar{\rho}u_1}$ ,  $w_{\bar{\rho}u_2}$ ,  $w_{\bar{\rho}u_3}$ , and  $w_{\bar{\rho}e}$  are propagation velocity of waves for density, 1 component of momentum, 2 component,

3 component, and energy, respectively. The propagation velocities are considered fluid velocity. The solution of eq. (18) is

$$\begin{aligned}
 u_j(X_1 | x_j, t) = & \exp(-a_{1j} \tau_{1j}) \\
 & [ \varepsilon_{1j}(X_1, x_j) \delta(\zeta_j - w_{1j} \tau_{1j}) \\
 & + \frac{a_{1j}}{2w_{1j}} \left\{ I_0 \left( a_{1j} \tau_{1j} \sqrt{1 - \frac{\zeta_j^2}{w_{1j}^2 \tau_{1j}^2}} \right) \right. \\
 & + \frac{1 + \frac{\zeta_j \varepsilon_{1j}(X_1, x_j)}{w_{1j} \tau_{1j}}}{\sqrt{1 - \frac{\zeta_j^2}{w_{1j}^2 \tau_{1j}^2}}} I_1 \left( a_{1j} \tau_{1j} \sqrt{1 - \frac{\zeta_j^2}{w_{1j}^2 \tau_{1j}^2}} \right) \left. \right\} ] \quad (21)
 \end{aligned}$$

for  $j$ th element of vector  $\bar{U}$ .  $X_1$  is the coordinate of direction chosen.  $\zeta_j = X_j - x_j$ ,  $\tau_{1j} = t - t_0$ , and  $w_{1j}$  is the  $j$ th element of diagonal part of matrix (20).  $I_i(y)$  is  $i$ th order Bessel's function of argument  $y$ .



Initial conditions

$$u_j(X_1 | x_j, t_0) = \varepsilon_{1j}(X_1, x_j) \delta(X_j - x_j) \quad (22)$$

and

$$\varepsilon_{1j}(X_1, x_j) = P\{u_j(X_j | x_j, t_0) | X_j(x_j, t_0) = x_j\}, \quad (23)$$

where the variable  $\varepsilon_{1j}(X_1, x_j)$  is the probability distribution of  $u_j(X_1 | x_1, t)$  at coordinate  $X_1$  and time  $t$  under the conditions of  $u_j(x_1 | x_1, t)$  and  $u_1(x_j | x_j, t_0)$  at time  $t$  and  $t_0$ .

### III. Statistical mechanics of fluid turbulence

In section II solution of spatial average of transport equation is expressed. Eddy transport coefficients are obtained in this section.

The average  $\langle X_i \rangle_j$  and the variance  $\langle (X_i - \langle X_i \rangle_j)^2 \rangle_j$  of  $X_j$  with respect to measure are defined as

$$\langle X_i \rangle_j \equiv \int dX_i (\bar{\rho} \tilde{u}_j(X_i)) X_i \quad (23)$$

and

$$\langle (X_i - \langle X_i \rangle_j)^2 \rangle_j = \int dx_i \left( \bar{\rho} \tilde{u}_j(X_i) \right) (X_i - \langle X_i \rangle_j)^2,$$

respectively. They are calculated with (24) the result (24) as

$$\langle X_i \rangle_j = x_i + \frac{w_{ij}^z \varepsilon_{ij}^z(x_i, x_j)}{2a_{ij}} (1 - \exp[-2a_{ij} \tau_{ij}]) \quad (25)$$

and

$$\begin{aligned} \langle (X_i - \langle X_i \rangle_j)^2 \rangle_j &= \frac{w_{ij}^z \tau_{ij}}{a_{ij}} \left[ 1 - \frac{1 - \exp[-2a_{ij} \tau_{ij}]}{2a_{ij} \tau_{ij}} \right] \\ &+ \varepsilon_{ij}^z(x_i, x_j) \frac{(1 - \exp[-2a_{ij} \tau_{ij}])^2}{4a_{ij} \tau_{ij}}. \end{aligned} \quad (26)$$

The coefficient of eddy viscosity is obtained as

$$\begin{aligned} D_{ij} &= \frac{\langle (X_i - \langle X_i \rangle_j)^2 \rangle_j}{2\tau_{ij}} = \frac{w_{ij}^z}{2a_{ij}} \left[ 1 - \frac{\exp[-2a_{ij} \tau_{ij}]}{2a_{ij} \tau_{ij}} \right] \\ &+ \varepsilon_{ij}^z(x_i, x_j) \frac{(1 - \exp[-2a_{ij} \tau_{ij}])^2}{4a_{ij} \tau_{ij}}. \end{aligned} \quad (27)$$

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### IV. Conclusions.

The eddy transport coefficient is calculated as

$$D_{\tilde{y}} = \frac{w_{\tilde{y}}^2}{2a_{\tilde{y}}} \left[ 1 - \frac{\exp[-2a_{\tilde{y}}\tau_{\tilde{y}}]}{2a_{\tilde{y}}\tau_{\tilde{y}}} + \varepsilon_{\tilde{y}}^2(x_i, x_j) \frac{(1 - \exp[-2a_{\tilde{y}}\tau_{\tilde{y}}])^2}{4a_{\tilde{y}}\tau_{\tilde{y}}} \right] \quad (28)$$

and substituting the relation of  $\tau_{\tilde{y}} = L/w_{\tilde{y}}$

leads to the relation as

$$D_{\tilde{y}} = \frac{w_{\tilde{y}}^2}{2a_{\tilde{y}}} \left[ 1 - \frac{\exp[-2a_{\tilde{y}}L/w_{\tilde{y}}]}{2a_{\tilde{y}}L/w_{\tilde{y}}} + \varepsilon_{\tilde{y}}^2(L, L) \frac{(1 - \exp[-2a_{\tilde{y}}L/w_{\tilde{y}}])^2}{4a_{\tilde{y}}L/w_{\tilde{y}}} \right]. \quad (29)$$

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